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Identification of boundary planes in three-dimensional flows

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Abstract

Let $\vec{v} = \vec{v}(p,t)$ be the velocity field of a Newtonian fluid, $\vec{\omega} = \vec{\omega}(p,t)$ its vorticity field and (e_{ij}) its 2-covariant rate-of-strain tensor. In this paper we give a formulation to identify boundary planes in analytical and numerical three-dimensional flow fields. The proposed formulation is based on the calculation of the locus where $\sum_{i,j=1}^{3} v_i \omega_j e_{ij} = 0$ is verified.

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1. Introduction

Several authors have proposed techniques and definitions to detect vortical structures in flows. This kind of flow organization is of importance to analyze the structure and the properties of the flow because vortices are commonly responsible for mixing processes and for large rates of momentum and heat/mass transfer (see, for example, Hunt et al. [1], Chong et al. [2], Jeon and Hussain [3], Michard et al. [4], Cucitore et al. [5], Wu et al. [6], Haller [7], Roth and Peikert [8,9]).

All these techniques, although useful to study the structure and organization of flows, have virtues and problems. In this study we report another method to extract information of the flows proposing an analytic procedure to identify boundary planes in three-dimensional flow fields. We also present some examples to illustrate the proposed technique.

2. Boundary plane

Let be $\mathcal{F}$ a flow of a Newtonian fluid in $\mathbb{R}^3$ (oriented Euclidean space of dimension three), then we can consider the trio $(\vec{v}(p,t), \vec{\omega}(p,t), D_{(p,t)})$ formed by the smooth velocity vector field of $\mathcal{F}$, its vorticity field; $\mathrm{curl}(\vec{v})$; and its 2-covariant rate-of-strain tensor, respectively.

We designate a boundary plane of $\mathcal{F}$ a plane σ such that it separates regions of the flow without momentum exchange. Particularly, the properties of this plane can be stated as:

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1. There is no convective flux of momentum across σ. In other words, σ is tangent to the smooth velocity vector field $\vec{v} = \vec{v}(p, t)$ of the flow at any fixed time t.
2. There is no diffusion flux of momentum across σ, consequently, the shear stresses vanish on σ. In others words

$$\frac{d(\vec{v}(p + s\vec{n}))}{ds}(0) = \lambda\vec{n}, \tag{1}$$

with $p \in \sigma$ is any point of σ and $\vec{n}$ is a unit normal vector to σ.

That is to say:

A boundary plane is a stream plane on which the shear stresses vanish.

2.1. Reformulation

We can prove the following

Result 1. *Let $\vec{v}$ be a smooth vector field in a domain Ω of the oriented Euclidean space of dimension three. Let σ be a plane in Ω, such that at each point $a \in \sigma$, the vector $\vec{v}(a)$ is tangent to σ at a. Let $\vec{n}$ be a unit normal vector to σ. Let $\vec{\omega}$ and D be, respectively, the vorticity field and the rate-of-strain tensor field of the vector field $\vec{v}$. Let $\vec{v}^{\perp} = \vec{n} \times \vec{v}$. Let p be a point in σ such that*

$$\frac{d(\vec{v}(p + s\vec{n}))}{ds}(0) = \lambda\vec{n}. \tag{2}$$

Then

$$\begin{aligned} D_p\big(\vec{v}(p), \vec{\omega}(p)\big) &= \sum_{i,j=1}^{3} v_i \omega_j e_{ij}|_p = 0, \\ D_p\big(\vec{v}^{\perp}(p), \vec{\omega}(p)\big) &= \sum_{i,j=1}^{3} v_i^{\perp} \omega_j e_{ij}|_p = 0. \end{aligned} \tag{3}$$

The rate-of-strain tensor is an important and well-known quantity in fluid mechanics. In what follows, we review this concept algebraically to clarify the formulae of Result 1. Given the velocity vector field $\vec{v}$ on the space $\mathbb{R}^3$, we have the 2-covariant rate-of-strain tensor field D and we denote by e_{ij} the components of D on the affine frame $\{p; \vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Then, we can construct for each point p the tensor (i.e. the bilinear application):

$$\begin{aligned} &T_p\big(\mathbb{R}^3\big) \times T_p(\mathbb{R}^3) \xrightarrow{D_p} \mathbb{R}, \\ &(\vec{u}, \vec{w}) \longrightarrow D_p(\vec{u}, \vec{w}), \end{aligned} \tag{4}$$

where the values on the couples of the basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ are

$$D_p(\vec{a}_i, \vec{a}_j) = e_{ij}. \tag{5}$$

Therefore, the value on any couple $\vec{u}$, $\vec{w}$ of vectors at the point p with $\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$, $\vec{w} = w_1\vec{a}_1 + w_2\vec{a}_2 + w_3\vec{a}_3$ is

$$D_p(\vec{u}, \vec{w}) = (u_1, u_2, u_3) \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}_p \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \vec{u} D_p \vec{w}^t \in \mathbb{R}. \tag{6}$$

The proof of Result 1 is straightforward. It is enough to consider an orthonormal positively oriented basis $\{\vec{e}_1, \vec{e}_2, \vec{n} = \vec{e}_3\}$, and express the vector fields $\vec{v}$, $\vec{\omega}$, and the tensor field $e_{ij} = D(\vec{e}_i, \vec{e}_j)$ in this basis.

This result can also be formulated as: $\vec{v}$ and $\vec{\omega}$, on the boundary planes, are orthogonal not only with the ordinary Euclidean scalar product, but also with respect to the bilinear symmetric form D_p.

3. Identification boundary planes

The proof of Result 1 allows us to propose the following procedure to find boundary planes.

Given the flow $\mathcal{F}$, we have the trio $(\vec{v}(p,t), \vec{\omega}(p,t), D_{(p,t)})$:

We calculate the geometrical locus L of the points p where

$$D(\vec{v}, \vec{\omega}) = D(\vec{\omega} \times \vec{v}, \vec{\omega}) = 0. \tag{7}$$

Then, we consider the subset $\Pi \subset L$ such that $\vec{v}$ and $\vec{\omega}$ are tangent and orthonormal to the locus to Π, respectively. *Consequently, by Result* 1, *Π contains the boundary planes.*

At this point one can question if all the connected regions of the geometrical locus Π constitute a set of planes.

Of course the conditions of the procedure are local, then a priori, we cannot affirm that all the connected regions of Π are planes. However, we can affirm that, as a first approach to find and identify boundary planes, the proposed procedure is very effective, because we can demonstrate the following:

Result 2. *Let $\vec{v}$ be a smooth vector field in a domain Ω of the oriented Euclidean space of dimension three. Let $\vec{\omega}$ and D be, respectively, the vorticity field and the rate-of-strain tensor field of the vector field $\vec{v}$, and $\vec{v}^{\perp} = \vec{\omega} \times \vec{v} \neq \vec{0}$. Let σ be a geometrical locus of the points where*

$$D(\vec{v}, \vec{\omega}) = D(\vec{v}^{\perp}, \vec{\omega}) = 0, \tag{8}$$

and such that σ is a stream, connected and smooth surface with $\vec{\omega}$ orthogonal to σ. Then, σ is a developable surface, i.e. the Gauss curvature of σ is null.

Proof. *Let p be a point in σ. Let $\gamma : s \to \gamma(s)$ be the integral curve of the vector field $\vec{v}$, which satisfies $\gamma(0) = p$. We know that, at least for $|s|$ small enough, $\gamma(s)$ lies in σ.*

In addition Herrera [10] proved the following:

Let S be a stream surface in a flow. Let p be a point of a streamline on S with $q \neq 0$. Then we have the following equations

$$\begin{aligned}
\omega_{\parallel} &= -2q\tau_g - 2D_p\left(\frac{\vec{v}^{\perp}}{q}, \vec{N}\right),\\
\omega_{\perp} &= -2qk_n + 2D_p\left(\frac{\vec{v}}{q}, \vec{N}\right),\\
\omega_3 &= 2qk_g - 2D_p\left(\frac{\vec{v}}{q}, \frac{\vec{v}^{\perp}}{q}\right),
\end{aligned} \tag{9}$$

where τ_g, k_n and k_g are, respectively, the geodesic torsion, the normal curvature and the geodesic curvature of the streamline, and D_p is the 2-covariant rate-of-strain tensor at p.

In formulae (9) $\vec{v}^{\perp}$ is the tangent vector field on S such that $\{\frac{\vec{v}}{q}, \frac{\vec{v}^{\perp}}{q}, \vec{N}\}$ is an orthonormal, direct basis (positively oriented), $q = \|\vec{v}\| \neq 0$ is the velocity of the particle, $\omega_{\parallel}$, $\omega_{\perp}$ are the tangent components on S of the vorticity: $\omega_{\parallel}$ is the component parallel to the flow and $\omega_{\perp}$ is the perpendicular component to the flow in the direction of $\vec{v}^{\perp}$, ω_3 is the vertical component of the vorticity to S and $\vec{N}$ is the normal vector given by a parametrization of S.

Therefore with the hypothesis and Eqs. (9), we have:

$$\begin{aligned}
\omega_{\parallel} &= -2q\tau_g - 2D_p\left(\frac{\vec{v}^{\perp}}{q}, \vec{N}\right) \Rightarrow 0 = \tau_g\big(\gamma(0)\big),\\
\omega_{\perp} &= -2qk_n + 2D_p\left(\frac{\vec{v}}{q}, \vec{N}\right) \Rightarrow 0 = k_n\big(\gamma(0)\big),
\end{aligned} \tag{10}$$

where $\tau_g(\gamma(0)), k_n(\gamma(0))$ are, respectively, the geodesic torsion and the normal curvature of γ at p respect the surface σ. Then, according to the formulae of the trihedron of Darboux–Ribaucour, $\vec{N}$ is fixed along γ. Consequently the Gauss curvature of σ at p is zero. □

It is well known that a surface with null Gauss curvature is a developable surface formed by the union of planar regions with non-planar ruled regions. Then, we note that using the Ossian Bonnet's formula

$$\tau_g\big(\gamma(0)\big) = \big(k_1(p) - k_2(p)\big)\cos\alpha\sin\alpha, \tag{11}$$

where α is the angle between the direction of maximum normal curvature of σ at p and $\frac{d}{ds}\gamma(s)|_{s=0}$, and k_1, k_2 are the principal curvatures, it is easy to prove that the non-planar regions of σ are such that at any point $\vec{v}$ is aligned with the rulings.

For the formulae used in this result, the readers can see for example Do Carmo [11] or others books of Differential Geometry of curves and surfaces.

We also note that the vanishing of the two smooth functions $D(\vec{v}, \vec{\omega})$ and $D(\vec{\omega} \times \vec{v}, \vec{\omega})$ generally (when these two functions are functionally independent) defines a line rather a surface. But, as we have proven, the boundary planes are located in regions where these two functions vanish.

4. Examples

In this section we show some examples of the application of the procedure described above to identify boundary planes in steady laminar three-dimensional flows. Particularly, we have applied the procedure to analytical solutions of the linearized Navier–Stokes equations at the onset of thermal convection of an infinite fluid layer heated from below and to numerical results of supercritical Rayleigh–Bénard convection in a cubical enclosure to show the ability of the method for the identification of boundary planes in analytic and numerically simulated flow fields. In these examples of natural convection flows, the gravity vector, $\vec{g} = -g\vec{k}$, is perpendicular to the horizontal bottom and top boundaries of the flow, which are kept at constant but different temperatures, being the bottom temperature higher than the top one.

Chandrasekar [12] reported the velocity field, given in the following equations, for the hexagonal flow pattern at the onset of thermal convection in a fluid layer with stress-free bottom and top bottom boundaries, located at $z = 0$ and $z = 1$, respectively.

$$\begin{aligned}
v_1 &= -\frac{\sqrt{3}}{4}\cos(\pi z)\cos\left(\frac{2\pi}{3}y\right)\sin\left(\frac{2\pi\sqrt{3}}{3}x\right),\\
v_2 &= -\frac{1}{4}\cos(\pi z)\sin\left(\frac{2\pi}{3}y\right)\left[\cos\left(\frac{2\pi\sqrt{3}}{3}x\right) + 2\cos\left(\frac{2\pi}{3}y\right)\right],\\
v_3 &= \frac{1}{3}\sin(\pi z)\left[\cos\left(\frac{2\pi\sqrt{3}}{3}x + \frac{2\pi}{3}y\right) + \cos\left(\frac{2\pi\sqrt{3}}{3}x - \frac{2\pi}{3}y\right) + \cos\left(\frac{4\pi}{3}y\right)\right].
\end{aligned} \tag{12}$$

We observed that in this flow, and also in the following examples, the locus where the conditions of Result 2 are verified can be found using only the following conditions

$$\Phi = D\vec{v}, \vec{\omega}) = 0 \quad \text{with } \nabla\Phi \cdot \vec{v} = 0, \tag{13}$$

because of the high degree of symmetry of the velocity fields of these particular examples.

For this velocity field,

$$\begin{aligned}
\Phi = \frac{25\pi^2}{1152\sqrt{3}}&\cos^2(\pi z)\sin(\pi z)\sin\left(\frac{2\pi y}{3}\right)\left[\sin\left(\frac{2\pi x}{\sqrt{3}}\right) + \sin(2\sqrt{3}\pi x)\right.\\
&+ \sin\left(\frac{2\pi}{3}(\sqrt{3}x + 4y)\right) + \sin\left(\frac{2\pi}{3}(\sqrt{3}x - 4y)\right) + \sin\left(\frac{2\pi}{3}(\sqrt{3}x + 2y)\right)\\
&+ \sin\left(\frac{2\pi}{3}(\sqrt{3}x - 2y)\right) - \sin\left(\frac{2\pi}{3}(2\sqrt{3}x + y)\right) - \sin\left(\frac{2\pi}{3}(2\sqrt{3}x - y)\right)\\
&\left.- \sin\left(\frac{4\pi x}{\sqrt{3}} + 2\pi y\right) - \sin\left(\frac{4\pi x}{\sqrt{3}} - 2\pi y\right)\right].
\end{aligned} \tag{14}$$

In the ranges $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ and $0 \leqslant z \leqslant 1$, the conditions $\Phi = 0$ and $\nabla\Phi \cdot \vec{v} = 0$, are verified for this flow in the following sixteen planes

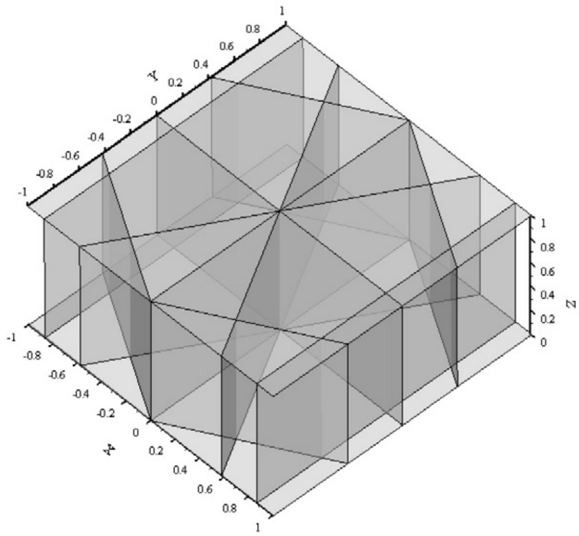

Fig. 1. Boundary planes of the hexagonal platform of convection.

$$
\begin{aligned}
&x=0, \quad x=\frac{\sqrt{3}}{2}, \quad x=-\frac{\sqrt{3}}{2},\\
&y=0, \quad y=\frac{1}{2}, \quad y=\frac{\sqrt{3}}{3}x, \quad y=-\frac{\sqrt{3}}{3}x, \quad y=\sqrt{3}x, \quad y=-\sqrt{3}x,\\
&y=\frac{\sqrt{3}}{3}x+1, \quad y=\frac{\sqrt{3}}{3}x-1, \quad y=-\frac{\sqrt{3}}{3}x+1, \quad y=-\frac{\sqrt{3}}{3}x-1,\\
&z=0, \quad z=\frac{1}{2}, \quad z=1.
\end{aligned}
\tag{15}
$$

The plane $z=\frac{1}{2}$ in the above list deserves special attention because is the only one in which $\nabla\Phi=\vec{0}$ and the alignment of the direction perpendicular to this plane with that of the velocity vector cannot be determined using $\nabla\Phi\cdot\vec{v}$. In this case the direction perpendicular to the plane can be determined taking the gradient of each of the components of $\nabla\Phi$. On this plane $\nabla(\nabla_x\Phi)=\vec{0}$, $\nabla(\nabla_y\Phi)=\vec{0}$, but $\nabla(\nabla_z\Phi)\neq\vec{0}$ and, thus, the perpendicular direction is $\nabla(\nabla_z\Phi)$, and the first condition is not verified on the plane $z=\frac{1}{2}$ because $\nabla(\nabla_z\Phi)\cdot\vec{v}\neq 0$. According to this, Fig. 1 shows the fifteen boundary planes of the hexagonal platform in the ranges $-1\leqslant x\leqslant 1$, $-1\leqslant y\leqslant 1$ and $0\leqslant z\leqslant 1$.

The velocity fields on the planes $z=\frac{1}{2}$ and $y=0$ are depicted in Figs. 2(a) and 2(b), respectively, in terms of the in-plane vector distribution and contours of the velocity component perpendicular to the corresponding plane (i.e. v_3 in Fig. 2(a)). Note that $y=0$ is a boundary plane. Consequently, $v_2=0$ and the contours are not shown in Fig. 2(b).

It should be noted that, as shown in this example, the proposed procedure allows the identification of symmetry planes, which satisfy the two conditions of the boundary planes.

The proposed procedure can be applied to detect boundary planes in numerically simulated flow fields. Pallares et al. [13] reported and described several stable flow topologies in steady laminar Rayleigh–Bénard convection in a cubical cavity. Fig. 3 shows that the isosurface, where $\Phi=0$ and $\nabla\Phi\cdot\vec{v}=0$ are verified, clearly identifies the boundary plane, located at $y=0.5$, of a single roll with its axis of rotation perpendicular to the lateral walls $y=0$ and $y=1$. The rolling motion of the flow is illustrated in this figure with pathlines. It should be noted that the conditions expressed in Result 2 are also satisfied on static walls, where $\vec{v}=0$. The isosurface plotted in Fig. 3 extends to the six walls of the cavity but this part of the isosurface has been hidden to facilitate the visualization of the pathlines and the interior border plane. The small portions of the isosurface that appear near the corners of the cavity can be attributed to the finite accuracy of the numerical calculation of Φ and $\nabla\Phi\cdot\vec{v}$ in the uniform computational grid of $41\times 41\times 41$ nodes used for the simulation of this flow.

Another stable flow structure in the cubical cavity is shown in Fig. 4. It consists of four rolls, each one with its axis perpendicular to one lateral wall of the cavity. This flow topology produces ascending motions near two diagonally

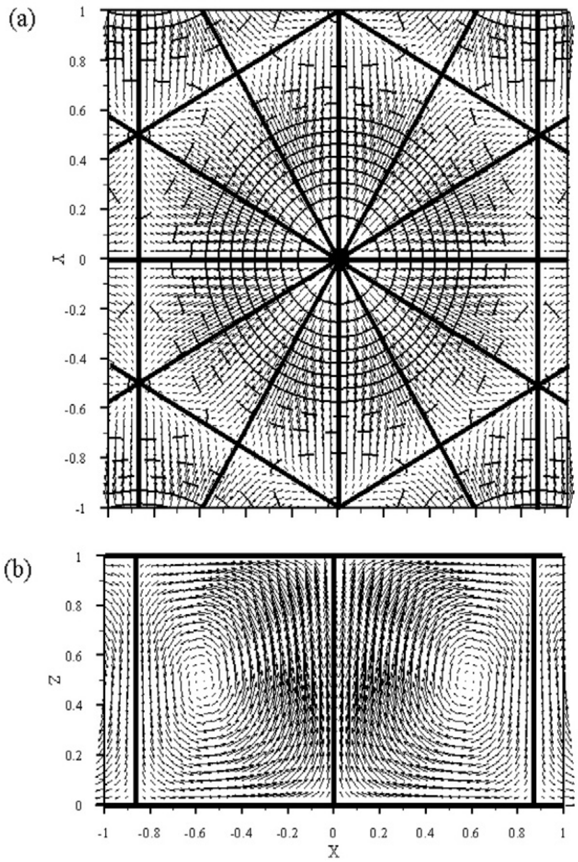

Fig. 2. Velocity field of the hexagonal cell in planes $z = 1/2$ (a) and $y = 0$ (b), in terms of the in-plane velocity vectors and contours of the velocity component perpendicular to the corresponding plane. Continuous line contours in Fig. 2(a) correspond to $u_3 > 0$ while dashed line contours correspond to $u_3 < 0$. The intersection of the boundary planes with the planes $z = 1/2$ and $y = 0$ are indicated with thick lines.

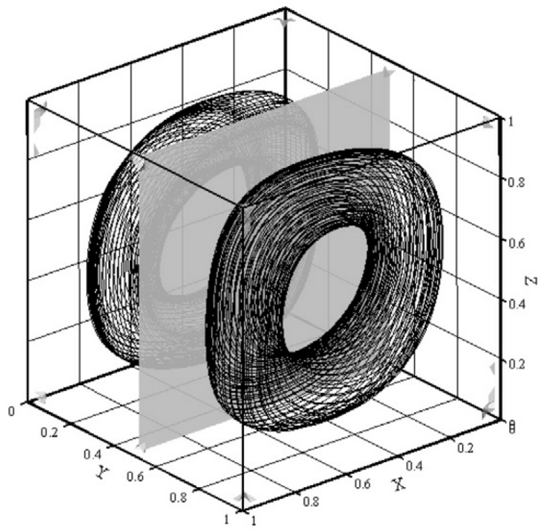

Fig. 3. Boundary plane of a single roll structure in a cubical cavity.

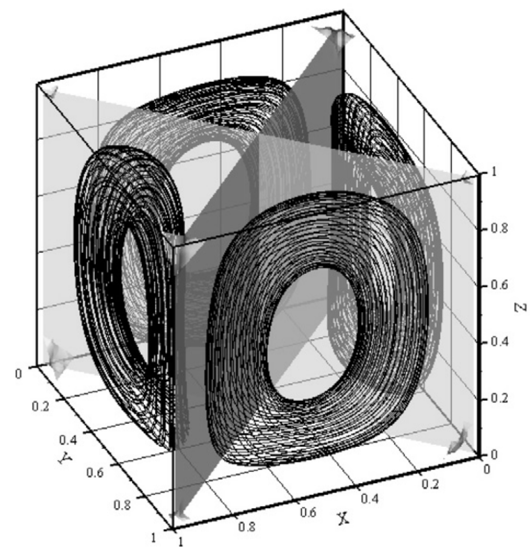

Fig. 4. Boundary planes of a four roll structure in a cubical cavity.

opposed vertical edges of the cavity while the flow descends near the other two diagonally opposed vertical edges. It can be seen that the two boundary of this flow structure, identified with the isosurface where $\Phi = 0$ and $\nabla\Phi \cdot \vec{v} = 0$ are verified, are located at $y = x$ and $y = -x + 1$.

5. Conclusions

We proposed an analytical method to find boundary planes in three-dimensional flow fields. The procedure is based on the calculation of the geometrical locus where the conditions $D(\vec{v}, \vec{\omega}) = D(\vec{\omega} \times \vec{v}, \vec{\omega}) = 0$. In this study we demonstrated the validity of these conditions on the boundary planes and we illustrated the application of the technique to identify boundary planes in closed solutions of the momentum equations and in numerically simulated flows.

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